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Relationship between two Calabi–Yau orbifolds arising as hyper–surfaces in a quotient of the same weighted projective space

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Abstract

In this article we consider a question: what is the relation between two Calabi-Yau manifolds of two different Berglund–Hubsch types if they appear as hyper–surfaces in the quotient of the same weighted projective space. We show that these manifolds are connected by a special change of coordinates, which we call the resonance transformation.

1. Introduction

Let $W_1^0(x_i)$ and $W_2^0(x_i)$ be two non–degenerate quasi-homogeneous polynomials satisfying

$$W_{1,2}^0(\lambda^{k_i} x_i) = \lambda^d W_{1,2}^0(x_i) \tag{1}$$

for $\lambda \in C^*$. Assume that $W_1^0(x_i)$ and $W_2^0(x_i)$ satisfy the Calabi–Yau condition

$$\sum_{i=1}^{5} k_i = d \tag{2}$$

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and belong to two different Berglund–Hubsch (BH) types [1]. They define two Calabi–Yau orbifolds (W_1^0, G) and (W_2^0, G), as hyper–surfaces in the quotient of a weighted projective space (WPS) P_{k_1,k_2,k_3,k_4,k_5} by some group G [1,2]. The group G must be admissible, which means that the group G is a subgroup of $Aut(W_1^0)$ and $Aut(W_2^0)$, and also contains the element j_w acting in projective coordinates as $j_w : x_j \to \exp(2\pi i k_j/d)x_j$. Since the BH polynomials $W_1^0(x_i)$ and $W_2^0(x_i)$ are invertible, using BHK mirror construction [1,2] for both of them, we obtain two mirror CY orbifolds $(\tilde{W}_1^0, \tilde{G})$ and $(\tilde{W}_2^0, \tilde{G})$, arising as hyper–surfaces in quotients of two different projective spaces, $P_{k_1(1),k_2(1),k_3(1),k_4(1),k_5(1)}$ and $P_{k_1(2),k_2(2),k_3(2),k_4(2),k_5(2)}$. The situation where this occurs is called the "multiple mirror phenomenon".

A natural question arises: what is the relationship between two BHK multiple mirrors $(\tilde{W}_1^0, \tilde{G})$ and $(\tilde{W}_2^0, \tilde{G})$? It has been proven in different ways by Shoemaker [3], Borisov [4], Kelly [5] and Clarke [6] that multiple mirrors of BHK are bi–rationally equivalent. Below we give a simple proof of this assertion. Also in [7], the periods of the non-vanishing holomorphic form of the multiple mirrors were calculated in a special case and it was found that they match. In ref. [8] it has been proven that the periods of the non vanishing holomorphic form coincide for all the cases when multiple BHK mirrors of the loop and chain types appear.

But there is one more question: what is the relation between two original Calabi–Yau (CY) orbifolds (W_1^0, G) and (W_2^0, G)? Since these varieties appear as hyper–surfaces in the quotient of the same weighted projective space, it is natural to think that each of them belongs to the complex structure moduli space of the other. However, a problem arises, how exactly are they related? In this paper, we prove that this relation is given by some change of coordinates, which we call the resonance transformation.

The plan of the paper is as follows. In section 2 we consider the relation between two singularities which are defined in the same quasi–homogenous space. In section 3 we recall the phenomenon of multiple BHC mirrors and give a simple proof of their birational equivalence [3–6]. In section 4, we formulate a hypothesis about the connection between the CY orbifolds themselves, whose multiple mirrors are bi–rationally equivalent. In section 5 we give an example of this connection for the case when two CY-manifolds of the Chain and Loop type appear in the same weighted projective space. In the appendix we give another example of Chain–Loop type correspondence.

2. Correspondence of singularities

We assume that we have two singularities, $W_1^0(x_i)$ and $W_2^0(x_i)$ which are defined in the same quasi–homogenous space. Namely,

$$W_r^0(\lambda^{k_i} x_i) = \lambda^d W_r^0(x_i), \tag{3}$$

for r=1, 2 and $\lambda \in C^*$ is any complex number. The integers k_i are called the weights. We assume that both singularities obey the same quasi–homogenous condition, and d is their degree. We assume that we have m variables, $i = 1, 2, \ldots, m$.

Our goal is to prove that $W_1^0(x_i)$ is the same singularity as $W_2^0(x_i)$, the result of the change of variables $x_i \to y_i$ and the result of the simultaneous deformation on the moduli space of W_2^0

$$W_1^0(x_i) = W_2^0(y_i) + \sum_{l=1}^{h} \phi_l e_l(y). \tag{4}$$

Here quasi-homogeneous polynomials $e_l(y)$ of degree d, belong to the ring of polynomials modulo the elements of the ideal generated by $\frac{\partial W_2^0}{\partial y_i}$ and ϕ_l are the coordinates on the moduli space of the second singularity.

The change of variables ('resonance transformation') is the most general polynomial transformation which is allowed by the quasi–homogenous condition. It is given by

$$x_i \to \sum_{n=1}^{n_i} a_{n,i} J_{n,i}(y), \tag{5}$$

where

$$J_{n,i} = \prod_{j=1}^{m} y_j^{P(n)_{ij}}. \tag{6}$$

And the non–negative integers $P(n)_{ij}$ are the solutions to the equation,

$$\sum_{j=1}^{m} P(n)_{ij} k_j = k_i. \tag{7}$$

We assume the most general solution to this equation and we denote by n_i the number of solutions corresponding to k_i. This is clearly the most general resonance transformation compatible with the quasi–homogeneity. The number of unknowns is given by

$$N = \sum_{i=1}^{m} n_i. \tag{8}$$

Now we get to the equations that must be satisfied by the resonance transformation. We wish to find a solution for the unknowns $a_{n,i}$. We denote by

$$W_1^0(x_i(y_j)) = W_2(y_i), \tag{9}$$

where $x_i(y_j)$ is the resonance transformation, eq. (5).

The equations that must be satisfied by $a_{n,i}$ are of two kinds. For simplicity, we assume for now that $W_2^0(y_i)$ is of Fermat type,

$$W_2^0(y_i) = \sum_{i=1}^{m} y_i^{A_i}, \tag{10}$$

where A_i is an integer compatible with the grading,

$$A_i = d/k_i, \tag{11}$$

where d is an integer and for all $i = 1, 2, \ldots, m$. Then the first equation is

$$C(W_2(y_i), y_i^{A_i}) = 1, \tag{12}$$

where $i = 1, 2, \ldots, m$ and where $C(a, b)$ denotes the coefficient of monomial b in the singularity a. The second equation is

$$C(W_2(y_i), \frac{\partial W_2^0}{\partial y_i} \prod_{j=1}^{m} y_j^{P(n)_{i,j}}) = 0 \tag{13}$$

for all $i, n = 1, 2, \ldots, m$ and $n \neq i$. Here we see again that the equations are in one-to-one correspondence with $\prod_{j=1}^{m} y_j^{P(n)_{ij}}$ and, in particular the number of equations is given by $N = \sum_i n_i$. Thus we have always the exact same number of equations as unknowns, eq. (5). This indicates that generically, there is a solution for these equations and we find that generically,

$$W_1(x_i(y_i)) = W_2(y_i) + \sum_l \phi_l e_l(y), \tag{14}$$

where $e_l(y)$ correspond to the moduli of the singularity W_2,

$$e_l(y) = \prod y_i^{m_i}, \text{ where } m_i \leq s_i - 2, \tag{15}$$

and integers m_i obey the quasi–homogeneity condition, eq. (7), and l runs over all moduli.

Let us discuss some examples. Consider the ‘loop’ singularity

$$W_1(x_i) = x_1^{n-1} x_2 + x_2^{n-1} x_1. \tag{16}$$

Here $m = 2$, $k_1 = k_2 = 1$ and $d = n$, which is an integer $n \geq 3$. Using the symmetry we may write the transformation, eq. (5), as

$$x_1 \to \alpha y_1 + \beta y_2, \quad x_2 \to \alpha y_2 + \beta y_1. \tag{17}$$

We assume that

$$W_2(y_i) = y_1^n + y_2^n, \tag{18}$$

which is of Fermat type. Here, α and β are the two unknowns. We define,

$$W = W_1(x_i(y_i)) \tag{19}$$

Then, the two equations become, according to eq. (12),

$$q_1 = C[W, y_1^n] - 1 = 0 \tag{20}$$
$$q_2 = C[W, y_1^{n-1} y_2] = 0 \tag{21}$$

We solve for $q_1 = q_2 = 0$, to get a and b. We denote the solution as

$$\{a \to a_0, b \to b_0\} \tag{22}$$

and finally,

$$\bar{W} = W(a_0, b_0) \tag{23}$$

which would be of the form W_1 up to moduli.

For $n = 3$ we find

$$a_0 = -1, \quad b_0 = \exp(i\pi/3) \tag{24}$$

as one of the solutions of eq. (12). Substituting we find

$$\bar{W} = W_2 = y_1^3 + y_2^3, \tag{25}$$

as expected, since there are no moduli in this case.

For $n = 4$ we find,

$$a_0 = -(1+i)/2 - \sqrt{2}, \quad b_0 = (1+i)/2 - \sqrt{2}, \tag{26}$$

where $i=\sqrt{-1}$, here. Substituting, we find,

$$\bar{W}=W(a_0,b_0)=y_1^4+y_2^4+6y_1^2y_2^2. \tag{27}$$

Which means that W_1 and W_2 are in the moduli space of one another.

Repeating the calculation for $n=5$ we find expressions for a_0 and b_0, which are approximately

$$a_0=0.541939-0.074175i, \quad b_0=0.0661011-1.17621i \tag{28}$$

and substituting we find

$$\bar{W}=y_1^5+y_2^5+(3+\sqrt{11}i)(y_1^3y_2^2+y_1^2y_2^3), \tag{29}$$

which is indeed of the form W_2 up to moduli. These are all the allowed moduli in these cases. It is noteworthy that all the solutions, for a_0 and b_0 give in this case the same expression for $\bar{W}$ up to $\pm$ on the square root. The same holds for $n=4$ but for $n>5$ it is not true anymore.

From these singularities we can build a Calabi–Yau manifold of complex dimension $n/3$. This we do by adding $n-2$ additional variables with weight 1 and power n,

$$W_3(x_i)=x_1^{n-1}x_2+x_2^{n-1}x_1+\sum_{r=3}^{n}x_r^n. \tag{30}$$

Then the equation $W_3(x_i)=0$ defines the n dimensional Calabi–Yau manifold. This manifold can be re–written by the transformation,

$$x_1\to a_0y_1+b_0y_2, \quad x_2\to a_0y_2+b_0y_1, \quad x_r\to y_r, \tag{31}$$

for $r=3,4,\ldots,n$. Then we see that the manifold $W_3=0$ becomes $W_4=0$ where

$$W_4=\sum_{r=1}^{n}y_r^n+\sum_{l=1}^{h}\phi_l e_l(y), \tag{32}$$

where $e_l(y)$ generate a deformation on the moduli space. This holds for any $n\geq 3$. Although, the explicit examples above are for $n=3,4,5$.

It is noteworthy that the manifold $W_4=0$ admits a realization as an solvable conformal field theory, deformed by some moduli. The CFT in this case is $(n-2)^n$, i.e., n copies of the $(n-2)$th $N=2$ minimal models. It is interesting that any singularity which has the correct homogeneity is isomorphic always to some product of minimal models plus moduli. For more details on this construction see refs. [11,12].

3. Bi–rationality of BHK multiple mirrors

In this section we give a simple proof that Calabi-Yau multiple BHK mirrors are bi–rationally equivalent [3–6]. We focus on the above mentioned case when the CY manifolds belong to two different BH types [1] and are defined as two hyper–surfaces in a weighted projective space P_{k_1,k_2,k_3,k_4,k_5}. Let the polynomials $W_1^0(x_i)$ and $W_2^0(x_i)$, be defined as

$$W_{1,2}^0(x_i)=\sum_{i=1}^{5}\prod_{j=1}^{5}x_j^{M_{ij}(1,2)} \tag{33}$$

and matrices $M(1)$ and $M(2)$ of two different BH types satisfy

$$\sum_{j=1}^{5} M_{ij}(1)k_j(1) = \sum_{j=1}^{5} M_{ij}(2)k_j(2) = d \tag{34}$$

where $d = \sum_{i=1}^{5} k_i$.

Then the two mirror CY manifolds are given by zeroes of polynomials $\tilde{W}_1^0$ and $\tilde{W}_2^0$ defined as

$$\tilde{W}_{1,2}^0(x_i) = \sum_{i=1}^{5} \prod_{j=1}^{5} x_j^{M_{ij}^T(1,2)}. \tag{35}$$

And $M_{ij}^T(1) = M_{ji}(1)$ and $M_{ij}^T(2) = M_{ji}(2)$ are matrices belonging two different BH types and satisfying

$$\sum_{j=1}^{5} M_{ij}^T(1)k_j(1) = \sum_{i=1}^{5} k_i(1) = d(1) \tag{36}$$

and

$$\sum_{j=1}^{5} M_{ij}^T(2)k_j(2) = \sum_{i=1}^{5} k_i(2) = d(2) \tag{37}$$

where $k_j(1)$ and $k_j(2)$ are mutually prime integers.

Let us look for the change of variables $x_i = x_i(y_1, ..., y_5)$ which will ensure the relation

$$\tilde{W}_1^0(x_i) = \tilde{W}_2^0(x_i(y_1, ..., y_5)). \tag{38}$$

The change of the variables $y_i = \prod_{j=1}^{5} x_j^{Q_{ij}}$ where matrix $Q = M(1)M(2)^{-1}$, guarantees this equality, as well as

$$\prod_{j=1}^{5} x_j^{M_{ji}(1)} = \prod_{j=1}^{5} y_j^{M_{ji}(2)}. \tag{39}$$

This is also easy to check using the property of the matrices $M(1)$ and $M(2)$, which follows from Calabi–Yau condition, namely from

$$\sum_{i,j=1}^{5} M_{ij}^{-1}(1,2) = 1, \tag{40}$$

that the change in the variables $x_i => \lambda^{k_i(1)} x_i$ implies $y_i => \lambda^{k_i(2)} y_i$.

Moreover, this change of variables also implies the equality

$$\prod_{j=1}^{5} x_j = \prod_{j=1}^{5} y_j, \tag{41}$$

which means that, in addition to the bi–rationality of the two varieties, the chiral rings associated with them are isomorphic.

4. The relation between two original Calabi–Yau orbifolds

In this section, we formulate a hypothesis about the connection between the CY orbifolds themselves, whose multiple mirrors are bi–rationally equivalent.

Namely, we assume that if two CY orbifolds, given by polynomials $W_1^0(x_i)$ and $W_2^0(x_i)$ of two different BH types, appear in the same weighted projective space P_{k_1,k_2,k_3,k_4,k_5}, then there exists a resonance transformation of the projective coordinates $x_i = f_i(y_1, ..., y_5)$ and a deformation of the complex structure, generated by monomials $e_l(y)$ of the same weight as $W_2^0(y)$ from the chiral ring, such that the following equation holds

$$W_1^0(f(y)) = W_2(y,\phi) = W_2^0(y) + \sum_{l=1}^{h_{21}} \phi_l e_l(y). \tag{42}$$

We will call this the 'Key equation'. The equality in the Key equation should not hold exactly, but in a weak sense, namely, modulo the sum of the elements of the ideal generated by derivatives of $W_2^0(y)$.

The resonance transformation $x_i = f_i(y_1, ..., y_5)$ means that

$$x_i = \sum_n a_{ni} J_{n,i}(y), \tag{43}$$

where $J_{n,i}(y)$, $n = 1, 2 \ldots$ are all possible quasi-homogeneous monomials which have the same weight as k_i. That is

$$J_{n,i}(y) = \prod_{j=1}^{5} y_j^{P(n)_{ij}}$$

and $\sum\limits_{j=1}^{5} P(n)_{ij} k_j = k_i$.

We emphasize that the numbers $P(n)_{ij}$ in the resonance transformation are assumed to be positive integers.

Also we define generators of the chiral rings as $e_l(y) = \prod\limits_{i=1}^{5} y_i^{S_{li}}$, where the integers S_{li} satisfy the equation $\sum\limits_{j=1}^{5} S_{li} k_j = d$ and some inequalities described in [9,10].

It is convenient to choose the generators of the ideal in the form

$$J_{n,i}(y) \partial_i W_2^0(y).$$

In the case when the weighted projective space admits the existence of more than one quasi-homogeneous polynomial $J_{n,i}(y)$ of weight k_i, the number of resonances increases.

The reason of this effect can be seen from Table 1 in [8], which gives the weights of all 111 cases P_{k_1,k_2,k_3,k_4,k_5} that admit the existence of two CY-manifolds, of type Loop and Chain, simultaneously. In each case we find that at least one of the weights k_m is equal to 1. It follows that the number of resonances, is the number positive integers m_j, that solve the equation $\sum\limits_j m_j k_j = k_i$.

5. Example

In this section we give an example of the above connection for the case when the two CY-manifolds are of the Chain and Loop type $W_1^0(x_i)$ and $W_2^0(x_i)$, which appear in weighted projective space $P_{23,17,41,27,1}$. Then polynomials of the chain and Loop types, correspondingly, are defined as

$$W_1^0(x_i) = x_1^4 x_2 + x_2^4 x_3 + x_3^2 x_4 + x_4^4 x_5 + x_5^{109} \tag{44}$$

and

$$W_2^0(x_i) = x_1^4 x_2 + x_2^4 x_3 + x_3^2 x_4 + x_4^4 x_5 + x_5^{86} x_1. \tag{45}$$

The resonance transformation of the projective coordinates $x_i = f_i(y_1, ..., y_5)$ looks in this case as follows

$$x_1 = a_3 y_1 + a_2 y_2 y_5^6 + a_1 y_5^{23}, \tag{46}$$

$$x_2 = a_5 y_2 + a_4 y_5^{17}, \tag{47}$$

$$x_3 = a_8 y_3 + a_{12} y_1 y_2 y_5 + a_{11} y_1 y_5^{18} + a_{10} y_2^2 y_5^7 + a_9 y_2 y_5^2 4 + a_7 y_4 y_5^{14} + a_6 y_5^{41}, \tag{48}$$

$$x_4 = a_{14} y_4 + a_{16} y_1 y_5^4 + a_{15} y_2 y_5^{10} + a_{13} y_5^{27}, \tag{49}$$

$$x_5 = a_{17} y_5. \tag{50}$$

Then from the Key equation (42) defined above we get the following five equations on the parameters of the resonance transformations a_n, $n = 1, ..., 17$

$$a_3^4 a_5 = a_5^4 a_8 = a_8^2 a_{14} = a_{14}^4 a_{17} = 1, \tag{51}$$

$$4a_1^3 a_3 a_4 + a_4^4 a_{11} + 2a_6 a_{11} a_{13} + a_6^2 a_{16} + 4a_{13}^3 a_{16} a_{17} = 1. \tag{52}$$

As we explain below, we can impose the following twelve additional equations on the a_n parameters,

$$4a_1^4 a_4 - 4a_2 a_3^3 a_4 - 4a_1 a_3^3 a_5 + 4a_4^4 a_6 + 4a_6^2 a_{13} - 2a_{11} a_{12} a_{16} + 4a_{13}^4 a_{17} - 4a_{15} a_{16}^3 a_{17} + 4a_{17}^{109} = 0, \tag{53}$$

$$16a_1^3 a_2 a_4 + 4a_1^4 a_5 - 4a_2 a_3^3 a_5 + 16a_4^3 a_5 a_6 + 4a_4^4 a_9 + 8a_6 a_9 a_{13} + 4a_6^2 a_{15} - a_{12}^2 a_{16} + 16a_{13}^3 a_{15} a_{17} = 0, \tag{54}$$

$$(a_5^4 - a_8 a_{14}) a_6 a_2^4 a_4 + 4a_1 a_2^3 a_5 + 4a_4 a_5^3 a_9 + 6a_4^2 a_5^2 a_{10} - a_7 a_8 a_{13} + a_{10}^2 a_{13} + 2a_9 a_{10} a_{15} + a_{15}^4 a_{17} = 0, \tag{55}$$

$$(a_5^4 - a_8 a_{14}) a_{11} + 4a_2^3 a_3 a_5 - a_7 a_8 a_{16} + a_{10}^2 a_{16} + 2a_{10} a_{12} a_{15} + 4a_4 a_5^3 a_{12} = 0, \tag{56}$$

$$a_2^4 a_5 + (a_5^4 - a_8 a_{14}) a_9 - a_7 a_8 a_{15} + a_{10}^2 a_{15} + 4a_4 a_5^3 a_{10} = 0, \tag{57}$$

$$4a_4 (a_3^4 - a_5^3 a_8) - 2a_8 a_{10} a_{15} + 4a_{16}^4 a_{17} = 0, \tag{58}$$

$$a_7 (a_5^4 - a_8 a_{14}) + a_{10}^2 a_{14} = 0, \tag{59}$$

$$4a_{13} (a_8^2 - a_{14}^3 a_{17}) - a_7^2 a_{14} = 0, \tag{60}$$

$$2a_{12} (a_5^4 - a_8 a_{14}) = 0, \tag{61}$$

$$4a_{15} (a_8^2 - a_{14}^3 a_{17}) = 0, \tag{62}$$

$$4a_{16} (a_8^2 - a_{14}^3 a_{17}) = 0, \tag{63}$$

$$2a_{10}(a_5^4 - a_8 a_{14}) = 0. \tag{64}$$

Equations (51) imply that a_3, a_5, a_8, a_{14} and a_{17} are non-zero.

It follows from equations (51) that (61)-(64) are fulfilled automatically. Taking this into account, from (60) and (59) we obtain that $a_7 = 0$ and $a_{10} = 0$. Then, it follows from (58) and (57) that $a_{16} = 0$ and $a_2 = 0$. Finally, from (56) we get that $a_4 a_{12} = 0$. Now we can choose the possible case $a_4 = 0$, which implies that $a_{15} = 0$.

After that, the non-vanishing parameters satisfy three equations:

$$a_6^2 a_{13} + a_{13}^4 a_{17} + a_{17}^{109} = a_1 a_3^3 a_5, \tag{65}$$

$$a_1^4 a_5 + 2a_6 a_9 a_{13} = 0, \tag{66}$$

$$2a_6 a_{11} a_{13} = 1, \tag{67}$$

along with the equations (51).

Thus we conclude that six parameters a_2, a_4, a_7, a_{10}, a_{12}, a_{16} vanish and other eleven are subject of seven equations.

As a result, we have obtained a 4-parameter family of solutions to the Key equation, which confirms our conjecture about the relationship between the two CY-manifolds.

Now we want to explain the origin of the twelve equations (51)–(64) which simplify the solution of the Key equation (42), but do not follow from it.

After substituting the resonance transformations (46)-(50) into the polynomial $W_1^0(x_i) = x_1^4 x_2 + x_2^4 x_3 + x_3^2 x_4 + x_4^4 x_5 + x_5^{109}$ we obtain the sum of monomials of the form $\prod_{j=1}^{5} y_j^{m_j}$, where sets of positive integers m_j satisfy the equation $\sum_{j=1}^{5} m_j k_j = d$.

The coefficients of those monomials that coincide with one of the five monomials in $W_2^0(y_i)$ must be equal to one, as follows from the Key equation, and this gives equations (51).

Coefficients of other monomials whose sets m_j satisfy the inequalities $m_1 < 4$, $m_2 < 4$, $m_3 < 2$, $m_4 < 4$, $m_5 < 86$ [10] can be left unchanged for now since these monomials belong to the chiral ring.

Finally, there is a third set of monomials, such as y_5^{109} which do not satisfy either of the above two definitions, and do not belong to the ideal, that is, they are not equal to a sum of monomials of the form $J_{ni}(y)\partial_i W_2^0(y)$. What to do with such monomials?

The answer is very simple. For example in the case y_5^{109} we just use the following equality

$$y_5^{109} = y_5^{23}\partial_1 W_2^0(y) - 4y_1^3 y_2 y_5^{23}. \tag{68}$$

The first term on the right-hand side of this equality belongs to the Ideal, and the second term coincides with one of the generators of the chiral ring. Therefore, the “unwanted” monomials simply change the coefficients of the twelve admissible monomials of the chiral ring.

We have used this fact to reduce the number of chiral ring monomials in the Key equation (42) by imposing equations (53)-(64). To get a general solution to the Key equation, we simply don’t have to impose these equations.

The equation for W_2 which includes the moduli is then seen to be,

$$W_2(y) = y(1)^4 y(2) + y(2)^4 y(3) + y(3)^2 y(4) + y(4)^4 y(5) + y(5)^{86} y(1) +$$
$$3y(4)y(5)^{82} - \frac{1}{4}y(2)^2\, y(5)^{75} + \left(\frac{9}{2} - 2i\right) y(1)y(2)y(5)^{69} - 2iy(3)\, y(5)^{68} -$$

$$y(2)y(4)y(5)^{65} - \frac{1}{4}y(1)^2y(5)^{63} + y(1)y(4)\ y(5)^{59} + 6y(4)^2y(5)^{55} -$$
$$iy(1)y(2)^2y(5)^{52} - iy(2)y(3)\ y(5)^{51} - \frac{1}{4}y(2)^2y(4)y(5)^{48} +$$
$$(6+i)y(1)^2y(2)y(5)^{46} + i\ y(1)y(3)y(5)^{45} + \left(\frac{1}{2} - 2i\right)y(1)y(2)y(4)\ y(5)^{42} -$$
$$\frac{3}{2}iy(3)y(4)y(5)^{41} - \frac{1}{4}y(1)^2y(4)\ y(5)^{36} + y(1)^2y(2)^2y(5)^{29} + \frac{15}{4}y(4)^3y(5)^{28} +$$
$$2y(1)y(2)\ y(3)y(5)^{28} - iy(1)y(2)^2y(4)y(5)^{25} - \frac{3}{4}iy(2)y(3)y(4)\ y(5)^{24} +$$
$$\frac{15}{4}y(1)^3y(2)y(5)^{23} + iy(1)^2y(2)y(4)\ y(5)^{19} + \frac{3}{4}iy(1)y(3)y(4)y(5)^{18} +$$
$$y(1)^2y(2)^2y(4)\ y(5)^2 - \frac{1}{4}y(1)y(2)^2y(3)y(5) + 2y(1)y(2)y(3)y(4)y(5)$$

6. Conclusion

In this work, we have shown that two Calabi-Yau manifolds of two different Berglund–Hubsch types that arise as hypersurfaces in an orbifold of the same weighted projective space are related by a special relation resonance transformation of coordinates.

Taking into account the correspondence [11,12], which plays an important role in superstring compactifications, between Calabi-Yau manifolds and $N = 2$ models of superconformal field theory, it would be interesting to understand what the relationship found between two CY-manifolds means for the two N=2 SCFT models corresponding to them.

Declaration of competing interest

No interest.

Data availability

Data will be made available on request.

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Appendix A

Let us give now another example o the map from chain to loop models. This is the example number 105 in the [8] list. Here the weightes are

$$k_i = \{28, 13, 21, 62, 1\} \tag{69}$$

and their sum $d = 125$ is the degree of quasi–homogenious polynomials W_1^0 and W_2^0. The chain Calabi–Yau model is given by $W_1 = 0$, where

$$W_1^0 = x_1^4 x_2 + x_2^8 x_3 + x_3^3 x_4 + x_4^2 x_5 + x_5^{125}. \tag{70}$$

The loop model is given by the manifold $W_2 = 0$ where

$$W_2^0 = y_1^4 y_2 + y_2^8 y_3 + y_3^3 y_4 + y_4^2 y_5 + y_5^{97} y_1. \tag{71}$$

Both are defined in the weighted projective space given by the weights k_i and d.

The transformation that takes us from W_1^0 to W_2 is the most general transformation respecting the weights. It is given by,

$$\begin{aligned}
&x(1) \to a(1)y(5)^{28} + a(3)y(2)y(5)^{15} + a(2)y(3)y(5)^7 + a(4)\ y(2)^2 y(5)^2 + a(5)y(1),\\
&x(2) \to a(6)y(5)^{13} + a(7)y(2),\\
&x(3) \to a(8)\ y(5)^{21} + a(10)y(2)y(5)^8 + a(9)y(3),\\
&x(4) \to a(11)y(5)^{62} + a(15)y(2)\ y(5)^{49} + a(13)y(3)y(5)^{41} + a(18)y(2)^2 y(5)^{36} +\\
&a(23)y(1)\ y(5)^{34} + a(16)y(2)y(3)y(5)^{28} + a(20)y(2)^3 y(5)^{23} + a(25)y(1)y(2)\ y(5)^{21} +\\
&a(14)y(3)^2 y(5)^{20} + a(19)y(2)^2 y(3)y(5)^{15} + a(24)y(1)y(3)\ y(5)^{13} + a(22)y(2)^4 y(5)^{10} +\\
&a(27)y(1)y(2)^2 y(5)^8 + a(17)y(2)y(3)^2\ y(5)^7 + a(28)y(1)^2 y(5)^6 +\\
&a(21)y(2)^3 y(3)y(5)^2 + a(26)y(1)y(2)\ y(3) + a(12)y(4),\\
&x(5) \to a(29)y(5),
\end{aligned}$$

where for convenience we denote by $y(w)$ and $a(q)$, y_w and a_q respectively. There are 29 unknowns which we denoted by $a(q), q = 1, 2, \ldots, 29$.

We denote the polynomial appearing above as $V_{r,j}$, i.e.,

$$x_r \to \sum_{j=n_{j-1}+1}^{n_j} a(j) V_{r,j}, \tag{72}$$

where n_j is the number of elements in the equation above, and $n_0 = 0$. Here, $n_j = \{5, 7, 10, 28, 29\}$. For example, $V_{1,j} = \{y(5)^{28}, y(2)y(5)^{15}, y(3)y(5)^7, y(2)^2 y(5)^2, y(1)\}_j$, $j = 1, 2, 3, 4, 5$.

Now, we wish to find out the equations obeyed by the parameters $a(j)$. These are the imposition of the vanishing of the ideal $\partial_r W$, multiplied by any of the $V_{r,j}$ polynomial. Suppose that our manifold is given by the 'loop',

$$W_2^0 = \sum_{r=1}^{5} y_r^{A_r} y_{r+1}, \tag{73}$$

where we identify $y_6 = y_1$. Also, $d = \sum_{r=1}^5 A_r$ is the degree of homogeneity, which is the Calabi–Yau condition. The 'chain' W_1^0 is given by

$$W_1^0 = \sum_{r=1}^{5} x_r^{A_r} x_{r+1}, \tag{74}$$

with the identification $x_6 = 1$ and $A_5 = d$. Here A defines the theory. For the present example $A = \{4, 8, 3, 2, 97\}$. We also define the vectors

$$f_r = \{y_5^{A_5}, y_1^{A_1}, y_2^{A_2}, y_3^{A_3}, y_4^{A_4}\} \tag{75}$$

and

$$s_r = y_r^{A_r-1} y_{r+1} \tag{76}$$

where we define $y_6 = y_1$. We define

$$W_2 = W_1^0(x(y)), \tag{77}$$

and $x(y))$ is the transformation eq. (72). Then, the equations obeyed by the a's are given by

$$eq(j) = C[W_2, V_{r,j} f_r A_r] - C[W_2, s_r V_{r,j}] = 0, \tag{78}$$

where $C[P, Q]$ is the coefficient of the monomial Q in the polynomial P. Here $j = 1, 2, \ldots, n_5 - 5$. In $V_{r,j}$ we omit the monomial y_r and we have the equations,

$$eq(j) = C[W_2, y_r^{A_r} y_{r+1}] \tag{79}$$

where $y_6 = y_1$ and j is $n_5 - 4, \ldots, n_5$. This gives all the equations. These equations implement the vanishing of the ideal. We omit the explicit general form of the equations for the sake of brevity. However, one of the many solutions of these equations is given by,

$$a(5) \to 1, a(7) \to 1, a(9) \to 1, a(12) \to 1, a(29) \to 1, a(1) \to \frac{1}{2^{2/3}},$$
$$a(6) \to 1, a(11) \to 1, a(3) \to \frac{1}{8}\left(16 - 3\sqrt[3]{2}\right), a(13) \to -\frac{1}{2},$$
$$a(15) \to -\frac{1}{8\ 2^{2/3}}, a(18) \to \frac{1}{512}\left(640 - 12\sqrt[3]{2} - 1537\ 2^{2/3}\right)$$

The rest of the variables are zero.

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